Honors Multivariable Calculus : : Class 12

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Definition 1. $f : D \to \mathbb{R}^m$ and $D \in R^n$ as an open set. We say that f is C^1 on D if all of f-s partial derivatives (first order) exist and are continuous on D.

Recall, f, g is $\mathbb{R} \to \mathbb{R}$ and if

$$h(x) = g(f(x))$$

then $h'(a) = g'(f(a)) \cdot f'(a)$ For small changes we can show,

$$\Delta x_1 = f'(a)\mathrm{d}x$$

$$\Delta x_2 = g'(f(a))\Delta x_1$$

I am thinking about using the linear operator twice.

212 Version of Chain rule is

$$z = \text{function of } u, v$$

$u, v =$ function of x, y

$$(u, v) = f(x, y)$$

$$z = g(u, v) = g(f(x, y))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$z = g(u, v)$$

$$(u, v) = f(x, y)$$

$$\mathrm{d}g_{(u_0,v_0)} = 1 \times 2 \text{ matrix } \mathrm{d}f_{x_0,y_0} = 2 \times 2 \text{ matrix}$$

The 1×2 matrix is $\begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$, and the 2×2 above is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$z = h(x, y) = g(f(x, y))$$

$$\mathrm{d}h_{(x_0,y_0)} = \mathrm{d}g_{(u_0,v_0)} \cdot \mathrm{d}f_{(x_0,y_0)}$$

A Good Beispiel for this

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 (truly differentiable). Suppose $z = f(x, y)$ and

$$\frac{\partial z}{\partial x}(5, 5) = 2$$

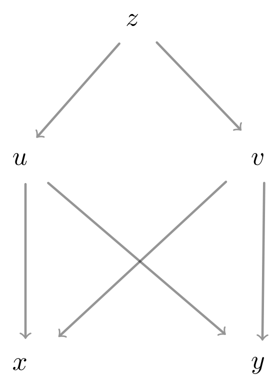

Figure 1: Map of chain rule for 212

$$\frac{\partial z}{\partial y}(5,5) = 4$$

We know how z changes now. Let's change to polar coordinates. What is $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$?

$$(x, y) = p(r, \theta)$$

We know that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

The line in polar makes a coordinate $(5\sqrt{2}, \frac{\pi}{4})$. So,

$$\begin{pmatrix} \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \mathrm{d}f_{(5,5)} \cdot \mathrm{d}p_{(5\sqrt{2}, \frac{\pi}{4})}$$

$$= \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -5 \\ \frac{\sqrt{2}}{2} & 5 \end{pmatrix} = d(f \cdot p)_{(5\sqrt{2}, \pi/4)}$$

The Gradient

Take $f : \mathbb{R}^n \to \mathbb{R}$ here if f is differentiable at $\vec{a}$ then $df_{\vec{a}}$ is a

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix}$$

$$D_{\vec{v}} f(\vec{a}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \vec{v}$$

$$D_{\vec{v}} f(\vec{a}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \cdot \vec{v}$$

So we just did a $df_{\vec{a}}^t$ transpose of the original that is in $\mathbb{R}^n$.

Definition 2. If $f : \mathbb{R}^n \to \mathbb{R}$ then the gradient of f at $\vec{a}$ at $\vec{a}$ in $\mathbb{R}^n$ is

$$\nabla f(\vec{a}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

If f is differentiable at $\vec{a}$ then $D_{\vec{v}} f(\vec{a})$ is $\nabla f(\vec{a}) \cdot \vec{v}$ for all unit vectors of $\vec{v}$.

$$\mid \nabla(f(\vec{a})) \mid \cdot \mid \hat{v} \mid \cos\theta = \mid \nabla f(\vec{a}) \mid \cos\theta$$

Using $\cos\theta$ at max $\mid \nabla f(\vec{a}) \mid$ is the maximum.

- The largest possible derivative
- Direction is where the derivative is largest
- Let $C = f(\vec{a})$ and then $X = f^{-1}(\{C\})$. Set of all point thats f maps to C. This is called "Leveled Hypersurface", eg

  $$f(x, y, z) = x^2 + y^2 + z^2$$
  $$f(1, 1, 1) = 3$$

  Then $f^{-1}(\{3\})$ level hypersurface for f is set of all points that becomes 3 so that is a sphere. If $p : \mathbb{R} \to \mathbb{R}^n$ is a curve (differentiable) with $p(t) \in X$ for all t and $p'(b) = \vec{a}$ then $f(p(t)) = C$ constantly.

  $$(f(p(t))' = 0$$

  So

  $$\mathrm{d}f_{p(b)} \cdot p'(b) = 0$$
  $$\nabla f(p(\vec{b})) \cdot p'(b) = 0$$
  $$\nabla f(\vec{a}) \cdot p'(b) = 0$$

  Gradient is perpendicular to any curve travelling along the surface on $\vec{a}$.