Honors Multivariable Calculus : : Class 11

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Partial derivatives of f at $\vec{a}$ with respect to x_i is

$$D_{e_i} f(\vec{a})$$

More common notation is $\frac{\partial f}{\partial x_i}(\vec{a})$ Sometimes you will see

$$f(x, y) = z$$

So

$$\frac{\partial z}{\partial x} = f_x$$

An example can be $f(x, y) = (\sin y + x^2 e^y, x + 2xy)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and we want $\frac{\partial f}{\partial x}$ at $(2, 0)$?

We can do this on the long way so by definition

$$\lim_{t \to 0} \frac{f(2, 0) + t\vec{e}_1 - f(2, 0)}{t}$$

$$\lim_{t \to 0} \frac{f(2 + t, 0) - f(2, 0)}{t}$$

$$\lim_{t \to 0} \frac{\begin{pmatrix} 0 + (2 + t)^2 \\ 2 + t + 2(2 + t) \cdot 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}}{t}$$

We have a simple single variable derivative.

$$\begin{pmatrix} \lim_{t \to 0} \frac{(2 + t)^2 - 4}{t} \\ \lim_{t \to 0} \frac{2 + t + 2(2 + t)0 - 2}{t} \end{pmatrix}$$

This just boils down into treating the individual components as individual derivatives.

$$\frac{\partial f}{\partial x}(2, 0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Just doing single variable we can find

$$\frac{\partial f}{\partial y}(2, 0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

These are directional derivative along $\vec{e}_1$ and $\vec{e}_2$.

$$\frac{\partial f}{\partial x}(2, 0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = D_{\vec{e}_1} f(2, 0)$$

$$\frac{\partial f}{\partial y}(2, 0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = D_{\vec{e}_2} f(2, 0)$$

If f is differentiable and $df_(2, 0)$ is represented by matrix M then

$$M = \begin{pmatrix} 4 & 5 \\ 1 & 4 \end{pmatrix}$$

$$df_{(2,0)}(\vec{e}_1) = M\vec{e}_1$$
$$df_{(2,0)}(\vec{e}_2) = M\vec{e}_2$$

If f is differentiable at $\vec{a}$ then Mx for $df_{\vec{a}}$ is

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{a}) & \dots & \frac{\partial f}{\partial x_n}(\vec{a}) \end{pmatrix}$$

Consider the vertical spanning of this matrix too. We have columns here beware!

If $f(\vec{a})$ is $(f_1(\vec{a}), f_2(\vec{a}), \dots, f_m(\vec{a}))$ if f is differentiable at $\vec{a}$ then

$$df_{\vec{a}} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \dfrac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix}$$

The ij entry is $\frac{\partial f_i}{\partial x_j}(\vec{a})$ So f is differentiable at $\vec{a}$, then this means f has directional derivatives in all directions of $\vec{a}$, then this also means f has partial derivatives along all n basis directions.

$$f(x, y) = \sqrt{|xy|}$$

Does not have directional derivatives (wait how why)

Partial derivatives around a region

Theorem 1. If $\frac{\partial f_i}{\partial x_j}$ all exist on some neighborhood of $\vec{a}$ and are continuous there (neighborhood basically means some small open set ball containing $\vec{a}$), the f is differentiable at $\vec{a}$.

Proof. Lemma: If $f : D \to \mathbb{R}^m$ and $f(\vec{a}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ then f is differentiable at $\vec{a}$ if and only if all f_i are differentiable at $\vec{a}$. (Left as homework)
For simplicity $n = 2$ by the lemma can just work with single coordinate function, so will take $f : \mathbb{R}^2 \to \mathbb{R}^1$. We will later find out how this generalizes to $\mathbb{R}^n$. We are assuming that $\frac{\partial f}{\partial x}$ is continuous at $\vec{a}$. I will write f_x for now, hence, f_x is continuous at $\vec{a}$ and also f_y. So we are trying to show that

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{a} + \vec{h}) - f(\vec{a}) - L(\vec{h})}{|\vec{h}|} = \vec{0}$$

$L(\vec{h})$ is what we think as the derivative. Here this $L(\vec{h})$ should be the matrix

$$L = \begin{pmatrix} f_x(\vec{a}) & f_y(\vec{a}) \end{pmatrix}$$

$$\text{LHS} = \frac{|f(\vec{a} + \vec{h}) - f(a_1 + h, a_2) + f(a_1 + h_1, a_2) - f(a_1, a_2) - L(h_1, h_2)}{\sqrt{h_1^2 + h_2^2}}$$

□

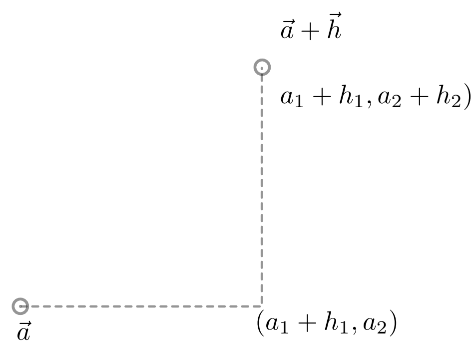

Figure 1: Showing continuity through partial derivatives