Honors Multivariable Calculus : : Class 10 (Woohoo!)

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Definition 1. $f : D \to \mathbb{R}^m$ is differentiable at $\vec{a}$ if for all linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{|\vec{h}|} = 0$$

If so L is $L = \mathrm{d}f_{\vec{a}}$ is derivative of f at $\vec{a}$.

For $f(x, y) = (x-1)^2 - (y+2)^2$ the derivative is going to be like $\mathrm{d}f_{(0,0)} \begin{pmatrix} p \\ q \end{pmatrix}$ is $-2p - 2q$

Proposition,

Theorem 1. If f is differentiable at $\vec{a}$ then f must be continuous at $\vec{a}$.

Proof. Let L be $\mathrm{d}f_{\vec{a}}$. Then we know that the limit as $h \to 0$, then

$$\frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{|\vec{h}|} = 0$$

We want to show (WTS) that

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

Using sequences would be easier, suppose that $\{\vec{x}_k\}$ is a sequence moving towards $\{\vec{a}\}$, let $\vec{h}_k = \vec{x}_k - \vec{a}$. We know that $\{\vec{h}_k\} \to 0$. So,

$$\frac{|f(\vec{a}+\vec{h}_k) - f(\vec{a}) - L(\vec{h}_k)|}{|\vec{h}_k|} \to 0$$

Multiply both sides with the denominator,

$$|\vec{h}_k| \frac{|f(\vec{a}+\vec{h}_k) - f(\vec{a}) - L(\vec{h}_k)|}{|\vec{h}_k|} \to 0$$

If norms go to zero then the vectors themselves go to zero.

$$\frac{|f(\vec{a}+\vec{h}_k) - f(\vec{a}) - L(\vec{h}_k)|}{|\vec{h}_k|} = 0$$

Then we have that

$$f(\vec{a}+\vec{h}_k) - f(\vec{a}) - L(\vec{h}_k) \to 0$$

For continuity

$$f(\vec{x}_k) - f(\vec{a}) \to \vec{0}$$

Hence

$$L(\vec{h}_k) \to 0$$

□

What does L is supposed to mean? Given $L = \mathrm{d}f_{\vec{a}}$. How should we think about $L(\vec{v})$? Of course $\vec{v} \neq 0$.

If

$$\vec{v} \approx 0$$

then

$$f(\vec{a} + \vec{v}) - f(\vec{a}) - L(\vec{v}) \approx 0$$

then $L(\vec{v}) \approx f(\vec{a} + \vec{v}) - f(\vec{a})$ and thus

$$L(\vec{v}) \approx \Delta f$$

if the change in the input of $\vec{v}$.

0.1 Thinking about $L(\vec{v})$

Another way to think about $L(\vec{v})$ while $\vec{v} \neq 0$. We want to think of the limit in the definition given above for a particular $\vec{h}$

$$\lim_{\vec{h} \to \vec{0}} \frac{|f(\vec{a} + \vec{h}) - f(\vec{a}) - L(\vec{h})|}{|\vec{h}|} = 0$$

Here $\vec{v} \in \mathbb{R}^n$. We can think of $\vec{h}$ as scalar multiples of $\vec{v}$. Consider

$$\vec{h} = t\vec{v}$$

We will presume what happens when $t \to 0$.

$$\lim_{t \to 0} \frac{|f(\vec{a} + \vec{v}t) - f(\vec{a}) - tL(\vec{v})|}{|\vec{v}||t|} = 0$$

What we get is,

$$\frac{1}{|\vec{v}|} \lim_{t \to 0} |\frac{f(\vec{a} + t\vec{v}) - f(\vec{a}) - tL(\vec{v})}{t}| = 0$$

$$\lim_{t \to 0} |\frac{f(\vec{a} + \vec{vt}) - f(\vec{a})}{t} - L(\vec{v})| = 0$$

This basically means

$$\lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} = L(\vec{v})$$

Think about the line that stretches along $\vec{v}$.

$$\lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} = L(\vec{v})$$

is called the Directional Derivative of f at $\vec{a}$ in the direction $\vec{v}$.

Theorem 2. Proposition: If $df_{\vec{a}}$ is L then $L(\vec{v})$ is the directional derivative at $\vec{a}$ along $\vec{v}$ and we say

$$D_{\vec{v}} f(\vec{a})$$

Prefer talking about $D_{\vec{v}} f(\vec{a})$ when $|\vec{v}| = 1$. Having a unit vector is professor Wangs preference.

$$f(x, y) = \frac{x^2 y}{x^4 + y^2}$$

It is zero $f(0, 0) = 0$ in such. Let's calculate it's directional derivative.

$$D_{\vec{v}} f(0, 0)$$

when $\vec{v} = \begin{pmatrix} p \\ q \end{pmatrix}$.

$$\lim_{t\to 0} \frac{f(\vec{0} + \vec{v}t) - f(\vec{0})}{t}$$

$$= \lim_{t\to 0} \frac{f(\begin{pmatrix} tp \\ tq \end{pmatrix}) - 0}{t} = \frac{\frac{t^2p^2tq}{t^4p^4+t^2q^2}}{t}$$

$$= \lim_{t\to 0} \frac{p^2q}{t^2p^2 + q^2} = \frac{p^2}{q}$$

We can't let $q \neq 0$.

We can't find L at this point to just simply pull a $L\vec{v}$ and get directional derivative. So

Definition 2. The partial derivative, the i-th one of f at $\vec{a}$ is just

$$D_{\vec{e}_i} f(\vec{a})$$

In a particular direction.

In general if we are at $\mathbb{R}^m$ then $D_{\vec{v}} f(\vec{a})$ lives in $\mathbb{R}^m$.