Honors Multivariable Calculus : : Class 07

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1 Differentiation

Definition 1. The single variable definition

$$f'(a) = \frac{\mathrm{d}f}{\mathrm{d}x}_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Attempt a definition where $f : \mathbb{R}^n \to \mathbb{R}^m$, here,

$$f'(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a})}{\vec{h}}$$

How tf do we divide with a vector? This can't work come on! Sensible one can be

$$f'(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + \vec{h}) - f(\vec{a})}{|\vec{h}|}$$

If we tried to solve for $g(0)$ if $g(x) = |x|$, then $g'(0) = 1$. $g'(2)$ does not exist (you can try the calculation).

We now need a new way to define what a derivative is. [Whatever equation we have, one verification that can tell us its definitely incorrect, if not correct is that for $f : \mathbb{R}^n \to \mathbb{R}^m$, $n = 1$ and $m = 1$, then the normal definition appears].

Let's say $f : \mathbb{R} \to \mathbb{R}^m$, $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$, and each $f_i(x)$ is $f_i : \mathbb{R} \to \mathbb{R}$. So what we have is,

$$\begin{aligned}\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{(f_1(a+h), \ldots, f_m(a+h)) - (f_1(a), \ldots, f_m(a))}{h} \\ &= (f_1'(a), f_2'(a), f_3'(a), \ldots, f_m'(a))\end{aligned}$$

I love the Wolfram Manipulation the prof is showing us now which is

$$p(t) = \left(\frac{1}{4}t^3, t^2\right)$$

What we have in the display is not a graph, a graph is $\mathbb{R}^{m+n}$. The curve has a sharp point but it's in the parametric of t. But if we thought of $x = \frac{1}{4}t^3$ and $y = t^2$, then the derivative won't have a $\frac{dy}{dx}$ solution at 0. As, $\frac{dy}{dx} = 4 \cdot \frac{2}{3}(4x)^{-\frac{1}{3}}$.

If $p : \mathbb{R}^1 \to \mathbb{R}^m$, and if p is differentiable, $p'(a)$ is a vector in $\mathbb{R}^m$. If p is position, then p' is a velocity vector tangent to path or $\vec{0}$. Magnitude for $|\vec{v}| = |\vec{p}'|$ is speed.

$$\vec{p}' = \vec{v} : \mathbb{R}^1 \to \mathbb{R}^m$$

Now what is $\vec{v}'(t)$ (the size of the smile in profs face, if n, there always exists N such that $n \geq N$). In single variable calculus $\vec{a} = \vec{v}'$ is a number. What does $a < 0$ mean by the way? $a < 0$ and $v > 0$ means we are slowing down. $a < 0$ and $v < 0$ means we are speeding up, in the opposite direction.

$p : \mathbb{R}^1 \to \mathbb{R}^m$, $v = p'$, $a = v' : \mathbb{R} \to \mathbb{R}^m$. If $a(t) = \vec{0}\ \forall t$, velocity is constant.