Honors Multivariable Calculus : : Class 06

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[Whatever I write inside third brackets are my own weird thoughts and they might not be true]

1 Thinking about Open Sets

Definition 1. Given some $D \subset \mathbb{R}^n$, we say that a point $\vec{a}$ interior of D if there $\exists r > 0$ such that $B_r(\vec{a}) \subset D$.

This means the ball around $\vec{a}$ is entirely put inside D. You can go any direction and still be in your domain. Wherever you go, around $\vec{a}$, but you will stay in D. Only interior points of a function are where we are capable of being differentiated.

Definition 2. Proposition: $V \subset \mathbb{R}^n$ is closed iff V contains all of it's limit points.

Proof. Suppose V is closed. We need to show it contains all of it's limit points. Then let $U = \mathbb{R}^n \setminus V$. So we can show that U is open and that contains no limit point of V. We only have definition of S sets that are open.
Let $\vec{x} \in U$ and we want to show $\vec{x}$ is not a limit point of V. What do we know about $\vec{x}$ if U is open. We can draw a ball and the ball is always inside of U. We know $\forall r > 0$ such that $B_r(\vec{x}) \subset U$ so $B_r(\vec{x}) \cap V = \phi$
Proof on the reverse direction:
Suppose V is contains all its limit points. We want to show V is closed, beispielen we want to show $U := \mathbb{R}^n \setminus V$ is open.
Let $\vec{x}$ is in U. Then $\vec{x}$ is not a limit point in V. To negate it we need a limit point definition, that there exists an $r > 0$ such that $B_r(\vec{x}) \cap (V \setminus \{\vec{x}\})$. So $B_r(\vec{x}) \cap V = \phi$, so $B_r(\vec{x}) \subset U$, hence U is open. [Negate means what it not means to be a limit points.] $\square$

Definition 3. Proposition: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then for all $U \subset \mathbb{R}^m$, the $f^{-1}(u)$ is open in $\mathbb{R}^n$. f might not be a bijection and it is NOT an inverse function beware. Sidebar: if f is $f : A \to B$ and $U \subset B$, then $f^{-1}(u)$ is pre-image of U under f and that is also $\{a \in A : f(a) \in U\}$

Proof. Let f be continuous and $U \subset \mathbb{R}^m$ be open. We need to show $f^{-1}(u)$ is open, so let $x \in f^{-1}(u)$. Then $f(\vec{x}) \in U$. Well U is open. Since U is open, $\forall \epsilon > 0$ such that $B_\epsilon(f(\vec{x})) \subset U$. Since f is continuous in $\vec{x}$, we have the limit that works for continuity. So $\forall \delta > 0$ such that if $|\vec{y} - \vec{x}| < \delta$, then $f(\vec{y}) - f(\vec{x}) < \epsilon$. This means that $f(\vec{y})$ is inside the ϵ ball around $f(\vec{x})$ which is $\subset U$. So $\vec{y} \in f^{-1}(u)$. This shows,

$$B_\delta(\vec{x}) \subset f^{-1}(u)$$

[For it to be open we need to show the ball around $\vec{x}$ is existant]. $\square$

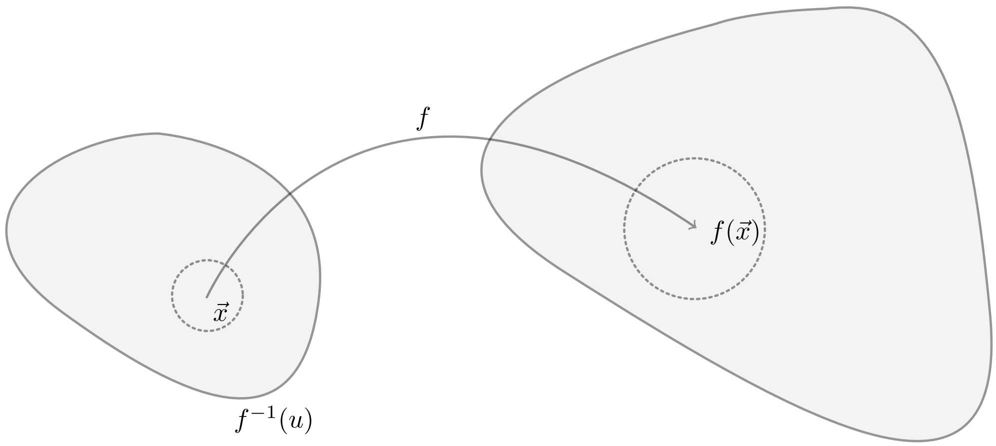

Figure 1: Preimage mapping from A to B

Definition 4. Proposition:

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

if $\forall U \subset \mathbb{R}^m$, we have $f^{-1}(u)$ is open in $\mathbb{R}^n$. Then f is continous.

Proof. Exercise lol □

Definition 5. $V \subset \mathbb{R}^n$ is bounded if $\exists r > 0$ such that $V \subset B_r(\vec{0})$.

Definition 6. * $V \subset \mathbb{R}^n$ is compact if V is closed and bounded.

We can have infinitely many $\vec{x}$ but still be careful about the boundary of this region. Any random $\vec{a}$ can have a ball that contains the whole set, this $\vec{a}$ can be anywhere. So we just pick up $\vec{0}$ to be the point of interest.

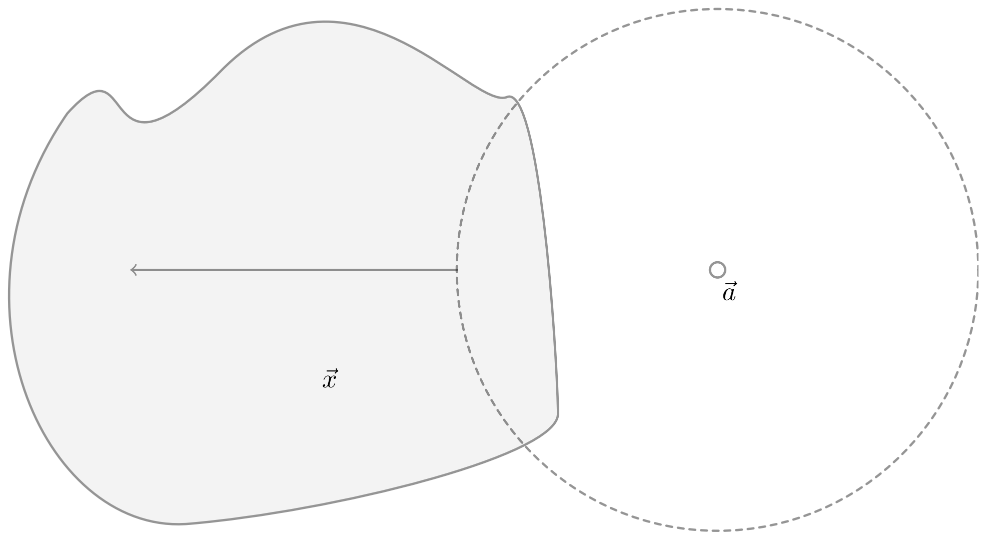

Figure 2: Beware, a set can have infinite points but the bound is kind of not infinite.