Honors Multivariable Calculus : : Class 05

January 19, 2024

Ahmed Saad Sabit, Rice University

We are about to move into continuity. Interestingly I worked on this last night. I will type things up when things get clear.

Definition 1.

$$f : D \to \mathbb{R}^m$$

Here $D \subset \mathbb{R}^n$, and $\vec{a}$ is limit of point in D.

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$$

implies for all sequence $\{\vec{x}_k\}$ in D,

$$\{\vec{x}_k\} \to \vec{a}; \vec{x}_k \neq \vec{a}$$

For that we get $\{f(\vec{x})\} \to \vec{L}$. The setup here will be that $\vec{a}$ will not be an isolated point and $\vec{a} \in D$.

$D \subset \mathbb{R}^n$ and $\vec{a}$ is a limit point of D if $\implies$

Definition 2. If $\vec{a} \in D$ and $\vec{a}$ is not a limit point of D, then we say that $\vec{a}$ is an isolated point of D. $\vec{a}$ is a limit point of D if $\forall r > 0$ there $\exists \vec{x} \in D \setminus \{\vec{a}\}$ with $|\vec{x} - \vec{a}| < r$.

$$D = [0, 1] \cup \{5\}$$

Here 5 is not a limit point of D.

For the definition of continuity, we can have $\vec{x} = \vec{a}$.

Definition 3. Continuity: If $f : D \to \mathbb{R}^m$ and $\vec{a} \in D$ and isn't isolated, then we say f is continuous at $\vec{a}$.

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

Definition 4. Proposition: $f : D \to \mathbb{R}^m$, and $g : E(\subset \mathbb{R}^m) \to \mathbb{R}^p$ and image$(f) \subset E$. If $\vec{a} \in D$ is not isolated, and f is continuous at $\vec{a}$ and g is continuous at $f(\vec{a})$, then $\implies$ $g \cdot f$ is continuous at $\vec{a}$.

Proof. We don't need epsilons-delta here. We will consider sequences that go to $\vec{a}$ and see what happens when we apply g on it. Consider $\{\vec{x}_k\}$ in D with $\{\vec{x}_k\} \to \vec{a}$ Since f is continuous at $\vec{a}$, we know,

$$\{f(\vec{x}_k)\} \to f(\vec{a})$$

Now do the exact same thing with g. Since g is continuous at $f(\vec{a})$, we conclude,

$$\{g(f(\vec{x}_k))\} \to g(f(\vec{a}))$$

□

This is not true for limits by the way. Single variable example, say $g(x)$ is 1 if $x = 0$. And $g(x)$ is 0 if $x \neq 0$. And $f(x)$ is 0.

So $\lim_{x \to 4} f(x) = 0$, and $\lim_{x \to 0} g(x) = 0$. But what about

$$\lim_{x \to 0} g(f(x)) = 1$$

There is no general rule for showing this limit being chained for each of the preceding term.

Let's work on an example,

$$\lim_{(x,y) \to (1,2)} x + y = 3$$

This function here $f(x)$ is $f : \mathbb{R}^2 \to \mathbb{R}$

Lets work on

$$f(x, y) = e^{x+y} \cos x$$

Split up in trees,

$$(x, y) \to x + y \to e^{x+y}$$

$$(x, y) \to x \to \cos x$$

$$(e^{x+y}, \cos x) \to e^{x+y} \cos x$$

Definition 5. $D \subset \mathbb{R}^n$ is an open set if $\forall \vec{a} \in D$ and $\exists r > 0$ such that $B_r(\vec{a}) \subset D$.

At any point in this D there is a neighborhood where D

You can draw an small ball literally anywhere in D. Open balls are open sets.

Definition 6. $D \subset R^n$ is closed if $\mathbb{R}^n \setminus D$ is an open set.

Closed is not the same thing as not open. Open is not the same thing as not closed set.

$$[1, 2]$$

$$(1, 2]$$

$$(1, 2)$$

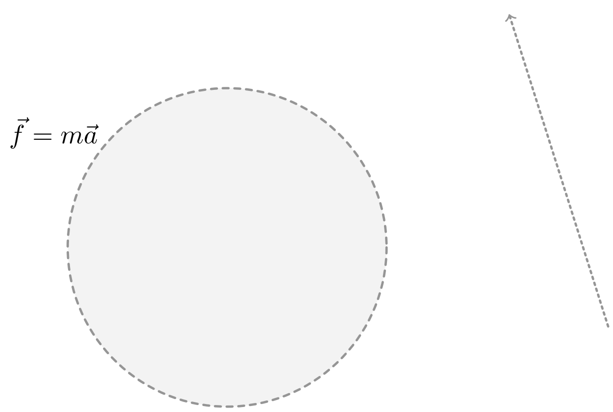

Figure 1: box figure