Honors Multivariable Calculus : : Class 04

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Forward direction was proved in last class.

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$$

if and only if $\forall$ sequence $\{x_k\}$ with $\vec{x}_k \to \vec{a}$ (but not equal) we have $\{f(\vec{x})\} \to \vec{L}$.

Proof. Let's prove the statement in reverse. Let's assume the limit is not $\vec{L}$. Suppose

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \neq \vec{L}$$

We will put our limit definition to work. Then $\forall \epsilon > 0$ such that $\forall \delta > 0$.
There exists $\vec{x}$ not equal ot $\vec{a}$, such that $|\vec{x} - \vec{a}| < \delta$ but $|f(\vec{x}) \to \vec{L} \geq \epsilon$
There is an ϵ around $\vec{L}$, that no matter what δ (which is around $\vec{x}$), there is an $\vec{x}$ is within this periphery of δ, but doing the mapping the $f(\vec{x})$ ends up being outside of the ϵ ball.
How to get a sequence out of this? ϵ is fixed but δ is anything above 0.
For any $k \in \mathbb{Z}^+$, letting δ as $\frac{1}{k}$, we can find $\vec{x}_k$ satisfying,

$$|\vec{x} - \vec{a}| < \frac{1}{k}$$

But $f(\vec{x}_k) - \vec{L}$ is still outside of ϵ ball. ($|f(\vec{x}_k) - \vec{L}| \geq \epsilon$).
Now we have sequence $\{\vec{x}_k\}$ which converges to $\vec{a}$ and we have the sequence $f(\vec{x}_k)$ of this thing which never gets within the ϵ of $\vec{L}$, so it doesn't converge. $\square$

Ruden Analysis for more, Davidson and Donnsig.

How can we describe $f(x) = \sqrt{x}$ if we want to solve for $\lim_{x\to 0} f(x)$. Domain in multivariable can look weird,

First we need to define what points are legitimate to take a limit off? This is the valid inputs that we can give our function to throw out $f(\vec{x})$.

Another examples, let's say our domain is $\mathbb{Z}$ in $\mathbb{R}$. Does it make sense to take a limit? Taking a limit towards 2 doesn't make sense because though 2 is in the domain it is we are not really intreested in $x = 2$.

Definition 1. If $D \subset \mathbb{R}^n$, we say that $\vec{a} \in \mathbb{R}^n$ is a limit point of D if you can go arbitrarily close to $\vec{a}$ without equalling $\vec{a}$ itself. Hence, if $\forall \epsilon > 0$ that $\exists \vec{x} \in D$ without $\vec{x} \neq \vec{a}$ but $|\vec{x} - \vec{a}| < \epsilon$.
Or you can say, for $\forall \epsilon > 0$, $B_\epsilon(\vec{a})$ contains a point in D that is not $\vec{a}$ itself.

$$B_\epsilon(\vec{a}) \cap (D \setminus \{\vec{a}\}) \neq \phi$$

Talk about limits if the domain is $\mathbb{Q}$ in $\mathbb{R}$. What does it mean to take a limit in such case?

Definition 2. Now

$$f : D \subset \mathbb{R}^n \to \mathbb{R}^m$$

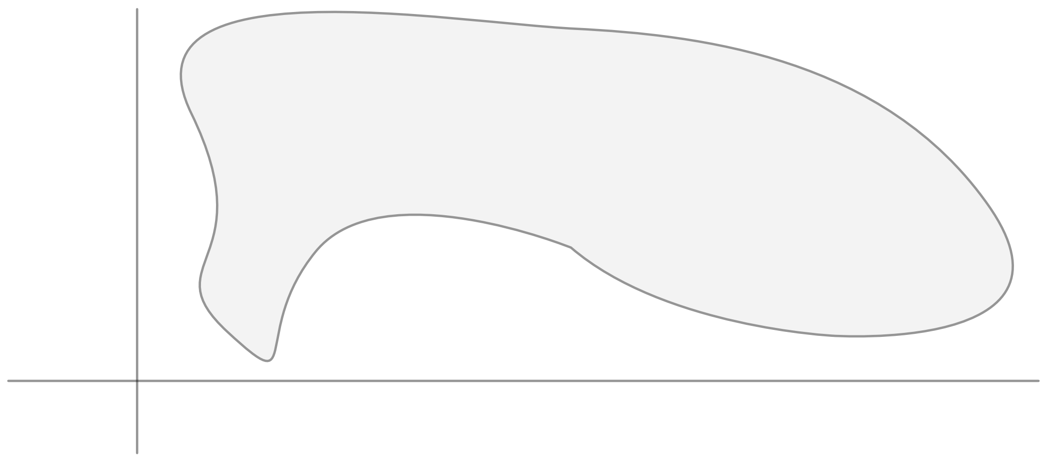

Figure 1: domain problem in multivariable calculus

and $\vec{a}$ is a limit point in D, then,

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$$

and $\vec{x} \subset D$, everything else is the same as the definition we did before. We are interested in the intersection of the ball and the domain.