Honors Multi Variable Calculus Class 02

January 10, 2024

Ahmed Saad Sabit

Note Because I have not made an example environment hence I will keep them in the problem environment.

Theorem 1. This

Definition 1. This

Problem 1. this

Solution. This □

1 Definition of the Limit $\epsilon - \delta$ rule

We had come across definition of limits pretty late. I remember reading this on the Complex Analysis by Serge Lang. We will use the formal definition of the limit.

Definition 2. Formal definition of limits goes like $\lim_{x\to a} f(x) = L$ for $\forall \epsilon > 0$ we have $\exists \delta > 0$ such that if $|x - a| < \delta$ where $x \neq a$, then $|f(x) - L| < \epsilon$.

$$\lim_{x\to a} f(x) = L; \forall \epsilon > 0, \exists \delta > 0,$$

We are currently talking about choosing the appropriate δ or ϵ to solve the problem.

Problem 2. $f : \mathbb{R} \to \mathbb{R}$, the Dirichlet goes like,

$$f(x) = 1$$

If x is rational.

$$f(x) = 0$$

If x is irrational. Now what is the $\lim_{x\to 0} f(x)$?

Solution. There is no limit. We can prove $f(x)$ is not 1 by saying the definition of limit is not working here. Negation would go like there exists $\epsilon > 0$ we have $\forall \delta > 0$, there exists x such that $|x - a| < \delta$ but $|f(x) - L| \geq \epsilon$. The professor goes on to choose $\epsilon = \frac{1}{2}$, no matter what we take δ to be, as long as $\delta > 0$, but we are trying to show that there is an x in this δ range where $f(x)$ is too far from L.
We can find some x which is irrational between 0 and δ. So $|x - 0| < \delta$ but $f(x) = 0$ so $|f(x) - 1| > \epsilon$. □

You can prove that two different limits at the same point. Interesting!

Problem 3.

$$\lim_{x\to 1} x^2 + 2x = 3$$

Proof. Let $\epsilon > 0$, we choose δ?
We have to do some scratch work before we decide what we want to be δ.
We want to get $|x^2 + 2x - 3| < \epsilon$. Where $|x - 1|$ is small enough. We need to figure out what small enough is.
We can do this,

$$|x^2 - 1 + 2x - 2| = |\,(x-1)\,(x+1) + 2(x-1)| \leq |(x-1)(x+1)| + |2(x-1)|$$

So both terms being $\frac{\epsilon}{2}$ gives us the total ϵ.Let's try something like both two terms of the last equation to be $< \frac{\epsilon}{2}$. This condition can be filled by,

$$2(x-1) < \frac{\epsilon}{2}$$

For which we have $x - 1 < \frac{\epsilon}{4}$.
For the first term we now just consider $|x - 1| < 1$, then $|x + 1| < 3$, and from here,

$$|(x-1)(x+1)| < 3|x-1|$$

And this will be $< \frac{\epsilon}{2}$ if $|x - 1| < \frac{\epsilon}{6}$. Now put all the three bounds for ϵ together, then, choosing $\delta = \min(1, \frac{\epsilon}{6})$.
We are flexible, we can just come up with any delta that works and that is enough.
So starting formally, $\epsilon > 0$, we choose $\delta = \min\left(\frac{\epsilon}{6}, 1\right)$, then if $|x - 1| < \delta$, then we know,

$$|x-1| < 1, 0 < x < 2, |x+1| < 3$$

So, $|(x-1)(x+1)| < \frac{\epsilon}{6} \cdot 3 = \frac{\epsilon}{2}$. Also $|2(x-1)| = 2|x-1| \leq 2 \cdot \frac{\epsilon}{6} \frac{\epsilon}{3} < \frac{\epsilon}{2}$.
Hence,

$$|x^2 + 2x - 3| \leq |(x-1)(x+1)| + |2(x-1)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

So, how does this extend to multivariable?

2 For multi variables?

What should happen for multi-variable function?

$$f : \mathbb{R}^2 \to \mathbb{R}$$

Or what should mean,

$$\lim_{(x,y)\to(1,4)} f(x,y) = 5$$

Intuitively, as (x, y) goes closer to $(1, 4)$, the output gets closer and closer to 5. The same definition holds, but we just want the distance of (x, y) from the limit instead of $|x - a|$ (where a is limit) to be lower than a bound δ.

You can still use something like,

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$$

It's still exactly the same thing, still!

Definition 3.

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L}$$

Definition for multiple variables is that

$$\forall\epsilon > 0, \exists\delta > 0$$

such that if

$$|\vec{x} - \vec{a}| < \delta$$

then,

$$|f(\vec{x}) - \vec{L}| < \epsilon$$

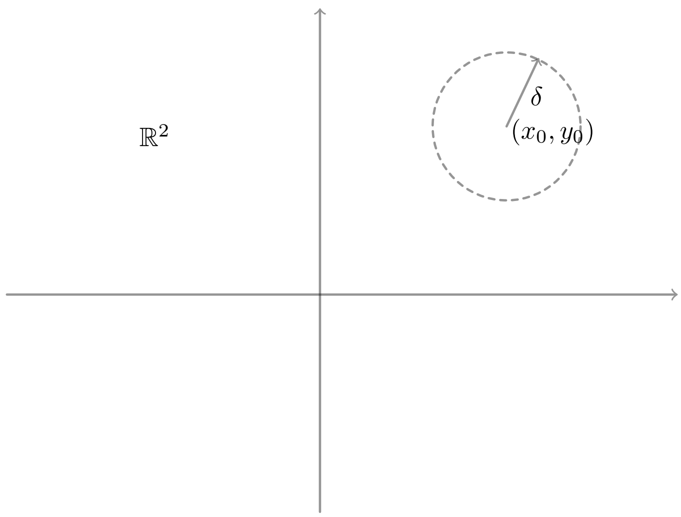

Figure 1: For the 2 Dimensional case the above figure is something like

Problem 4.

$$\lim_{x,y\to 1,2} x + y = 3$$

Proof. Let $\epsilon > 0$, we pick a δ. Scratch work,

$$|x + y - 3| < \epsilon$$

So,

$$|x - 1 + y - 2| \leq |x - 1| + |y - 2|$$

Set each terms to be lower than $\frac{\epsilon}{2}$. Then,

$$|x - 1 + y - 2| \leq \epsilon$$

We want the distance to be lower than δ, hence,

$$\sqrt{(x-1)^2 + (y-2)^2} < \delta$$

But using flexibly,

$$|x - 1| < \delta$$

And

$$|y - 2| < \delta$$

We want both of these terms to be lower than $\frac{\epsilon}{2}$, so just consider δ to be lower than $\frac{\epsilon}{2}$. Yay!
Formally, setting $|(x, y) - (1, 2)| < \delta = \frac{\epsilon}{2}$, we have set this. Hence both being smaller than

$$|x + y - 3| < \epsilon$$

□

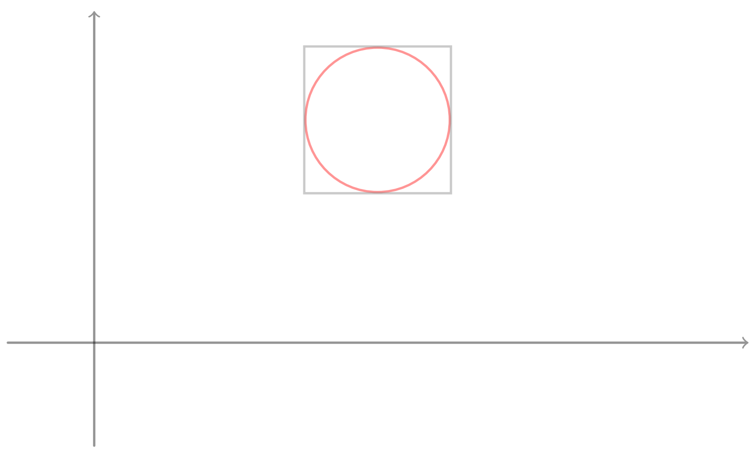

Figure 2: For the last problem the adjoined diagram