Honors Multivariable Calculus : : Class 01

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Note This text is being edited a lot later than the first class. Well, I didn't have any respectable system running to take notes in class that day.

Rigorous definition of a Limit is necessary

Single Variable Limit

Definition 1. Let there be a function $f : \mathbb{R} \to \mathbb{R}$ and let there exist a real number a. The number L is defined to be the limit L

$$\lim_{x \to a} f(x) = L$$

meaning that for all $\epsilon > 0$, there exists some $\delta > 0$ such that if

$$|x - a| < \delta$$

then

$$|f(x) - L| < \epsilon$$

A requirement is that $x \neq a$. Hence we say L is the limit of $f(x)$ if x „approaches" L.

We will later see if $x = a$ and $f(x) = L$ then it's definition of being continuous. We have defined closeness with a tricky way, we can pick ϵ to be anything above 0. This can be infinitely large, this can be infinitely close to 0. That's the beauty of this idea, it just nicely defines what's small without having to create weird abstractions.

I was thinking hard about limits when I was pretty young trying to define smallest limit. This brings the whole thing to perspective.

Multi Variable Limit

Multivariable Function

We can have single variable function like

$$f(2) = 3$$

But what is a vector function? Well this one takes input vectors and spits out another vector, not necessarily the same family of vector. Of course this is a linear map, or linear transformation, from either one space to another, or to itself.

$$f\left(\begin{bmatrix}2\\1\\4\\0\\1\end{bmatrix}\right) = \begin{pmatrix}7\\1\end{pmatrix}$$

This specific function takes members of $\mathbb{R}^5$ and gives out $\mathbb{R}^2$.

$$f : \mathbb{R}^5 \to \mathbb{R}^2$$

We can break it down like this

$$f(\mathbf{x}) = \begin{pmatrix}f_1(\mathbf{x})\\f_2(\mathbf{x})\end{pmatrix}$$

The good thing about f_1, f_2 is

$$f_1, f_2 : \mathbb{R}^5 \to \mathbb{R}^1$$

They take vectors and spit out simple numbers (scalars). The f_1 and f_2 can be something like

$$f_1(\mathbf{x}) = f_1(x_1, x_2, x_3, x_4, x_5) = 2x_1 - x_5 + x_3 = 7$$

Similarly, taking a list f_2 behaves

$$f_2(\mathbf{x}) = f_2(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 + x_4 - 6x_5 = 1$$

Multivariable limit is basically instead of dealing with $\mathbb{R}$, we deal with $\mathbb{R}^n$ where the inputs are vectors $\mathbf{v}$ and outputs are $\mathbb{R}^m$ members like $\mathbf{u}$. Concept of limit is basically broken down into components of the vectors.

Definition 2. Let there be a function $f : \mathbb{R}^n \to \mathbb{R}^m$ and points $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{L} \in \mathbb{R}^m$. The statement

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

meaning that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{a}| < \delta$ and $|f(\mathbf{x}) - \mathbf{L}| < \epsilon$ with the requirement $\mathbf{x} \neq \mathbf{a}$.

Theorem 1. Suppose $f : \mathbb{D} \to \mathbb{R}$ is a vector function of m dimension. This means,

$$f = (f_1, \ldots, f_m) \in \mathbb{R}^m$$

Let a member of $\mathbb{D}$ be $\mathbf{a}$. This $\mathbf{a}$ itself might be a vector if $D \subset \mathbb{R}^n$ with n dimensions. Now if $\mathbf{L} \in \mathbb{R}^m$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

If and Only If

$$\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$$

Where $i = 1, \ldots, m$.

Note The funny thing is we can also treat that like a theorem and prove it using single variable idea and introducing the idea of a norm $|\mathbf{v}|$. For this theorem, additionally it's important to consider $\mathbf{a}$ is a limit point. That means $\mathbf{a}$ isn't a boundary point (at the edge)or something and has infinite points around a small circle around. I will go indepth on this later using the idea of a Ball. For now consider $\mathbf{a}$ is well behaved.

Proof. Consider this

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

This means that there is a $\epsilon > 0$ such that

$$|f(\mathbf{x}) - \mathbf{L}| < \epsilon$$

From the definition of limit we say in the last section, well, the limit implies there must be a δ such that

$$|\mathbf{x} - \mathbf{a}| < \delta$$

So having ϵ we also must have δ follow the rule. Because it's „if and only if", it also implies if δ exists, then so as the bound ϵ.

Vector function limit implies Component Limit: let's say $|\mathbf{x} - \mathbf{a}| < \delta$. Then if component limit exists, we need a bound like.

$$|f_i(\mathbf{x}) - L_i| < \epsilon$$

We are going to prove this bound ϵ is existent.

Well we know that a component must be either equal or smaller than the vector it forms

$$|\alpha_i| \leq |\mathbf{A}|$$

Using this, and using the fact that the **Vector Function** already is under the bound ϵ,

$$|f_i(\mathbf{x}) - L_i| \leq |f(\mathbf{x}) - \mathbf{L}| < \epsilon$$

So we just showed

$$|f_i(\mathbf{x}) - L_i| \leq \epsilon$$

Component Limit implies Vector Limit: Given $|f_i(\mathbf{x}) - L_i| < \beta$ (some bound), then we want to show $|f(\mathbf{x}) - \mathbf{L}| < \epsilon$.

Now note that

$$|f(x) - \mathbf{L}| = \sqrt{\left[\sum_{i=1}^{m} |f_i(\mathbf{x}) - b_i|^2\right]}$$

This looks scary but this is just Pythagoras theorem lol.

Let me further break down the logic for what we are about to do now, so every $|f_i(\mathbf{x}) - L_i| < \beta$ is unique for every i-th value, so it's important we had called it β_i instead of just β. But I will assume out of all $\{\beta_i\}$, we pick the largest β so that it's bigger than every bound. Hence $\beta = \max(\{\beta_i\})$

Hence β being bigger than every term

$$|f(x) - \mathbf{L}| = \sqrt{\left[\sum_{i=1}^{m} |f_i(\mathbf{x}) - b_i|^2\right]} < \left[m\beta^2\right]^{\frac{1}{2}} = \varepsilon$$

Choosing $\varepsilon = \sqrt{m\beta^2}$ □